SPINORS, SPIN COEFFICIENTS AND LANCZOS POTENTIALS

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Abstract

It has been demonstrated, in a number of special situations, that the spin coefficients of a canonical spinor dyad can be used to define a Lanczos potential of the Weyl curvature spinor. In this paper we explore some of these potentials and show that they can be defined directly from the spinor dyad in a very simple way, but that the results do not generalize significantly, in any direct manner. A link to metric, asymmetric, curvature-free connections, which suggests a more natural relationship between the Lanczos potential and spin coefficients, is also considered.

1 INTRODUCTION

Throughout this paper we will use conventions and definitions from [16]. Let W_{ABCD} be an arbitrary symmetric spinor; a symmetric spinor $L_{ABCA'}$ is said to be a Lanczos spinor potential of W_{ABCD} if

$$W_{ABCD} = 2\nabla_{(A}{}^{A'} L_{BCD)A'}. \tag{1}$$

Illge [11] has shown that such a Lanczos potential always exists locally (of course this requires a four dimensional spacetime with a metric of Lorentz signature). The Lanczos spinor potential is essentially the spinor analogue of the Lanczos *tensor* potential [13] for which Bampi and Caviglia [5] have given an existence proof, by tensor methods in analytic, four dimensional spacetimes independent of metric signature. Note that no assumptions about the differential properties of W_{ABCD} are made. Illge has also shown that the solution of (1) is far from unique. For a recent summary of properties of the Lanczos potential, see [9]. Of particular interest is the case when $W_{ABCD} = \Psi_{ABCD}$ i.e., the Weyl curvature spinor. There exists no algorithm for finding Lanczos potentials in general spacetimes, but in certain special situations some algorithms have been found (see e.g., [3], [8], [9] and [15]).

Let (o^A, ι^A) be a spinor dyad, normalized so that $o_A\iota^A = 1$. It is then conventional to define the 8 dyad components of the Lanczos potential as

$$L_0 = L_{ABCA'}o^A o^B o^C o^{A'} \qquad L_4 = L_{ABCA'}o^A o^B o^C \iota^{A'}$$

$$
\begin{array}{ll}
L_1 = L_{ABCA'} o^A o^B \iota^C o^{A'} & L_5 = L_{ABCA'} o^A o^B \iota^C \iota^{A'} \\
L_2 = L_{ABCA'} o^A \iota^B \iota^C o^{A'} & L_6 = L_{ABCA'} o^A \iota^B \iota^C \iota^{A'} \\
L_3 = L_{ABCA'} \iota^A \iota^B \iota^C o^{A'} & L_7 = L_{ABCA'} \iota^A \iota^B \iota^C \iota^{A'} .
\end{array} \tag{2}
$$

Note that in [3] and [14], these Lanczos scalars are defined from the *tensor* version of the Lanczos potential; therefore the scalars used in [3] and [14] are defined using the opposite sign compared to our spinor definition.

Now, (1) can be written as 5 scalar equations in NP-formalism [3], [14]

$$
\begin{aligned}
\frac{1}{2}\Psi_0 &= \delta L_0 - DL_4 - (\bar{\alpha} + 3\beta - \bar{\pi})L_0 + 3\sigma L_1 + (3\varepsilon - \bar{\varepsilon} + \bar{\rho})L_4 - 3\kappa L_5 \\
2\Psi_1 &= 3\delta L_1 - 3DL_5 - \bar{\delta} L_4 + \Delta L_0 - (3\gamma + \bar{\gamma} + 3\mu - \bar{\mu})L_0 \\
&\quad -3(\bar{\alpha} + \beta - \bar{\pi} - \tau)L_1 + 6\sigma L_2 + (3\alpha - \bar{\beta} + 3\pi + \bar{\tau})L_4 \\
&\quad +3(\varepsilon - \bar{\varepsilon} + \bar{\rho} - \rho)L_5 - 6\kappa L_6 \\
\Psi_2 &= \delta L_2 - DL_6 - \bar{\delta} L_5 + \Delta L_1 - \nu L_0 - (2\mu - \bar{\mu} + \gamma + \bar{\gamma})L_1 \\
&\quad -(\bar{\alpha} - \beta - \bar{\pi} - 2\tau)L_2 + \sigma L_3 + \lambda L_4 + (\alpha - \bar{\beta} + 2\pi + \bar{\tau})L_5 \\
&\quad -(\varepsilon + \bar{\varepsilon} - \bar{\rho} + 2\rho)L_6 - \kappa L_7 \\
2\Psi_3 &= \delta L_3 - DL_7 - 3\bar{\delta} L_6 + 3\Delta L_2 - 6\nu L_1 + 3(\bar{\mu} - \mu + \gamma - \bar{\gamma})L_2 \\
&\quad -(\bar{\alpha} - 3\beta - 3\tau - \bar{\pi})L_3 + 6\lambda L_5 - 3(\alpha + \bar{\beta} - \bar{\tau} - \pi)L_6 \\
&\quad -(3\varepsilon + \bar{\varepsilon} - \bar{\rho} + 3\rho)L_7 \\
\frac{1}{2}\Psi_4 &= \Delta L_3 - \bar{\delta} L_7 - 3\nu L_2 + (\bar{\mu} + 3\gamma - \bar{\gamma})L_3 + 3\lambda L_6 - (3\alpha + \bar{\beta} - \bar{\tau})L_7
\end{aligned} \tag{3}
$$

for the Lanczos scalars; these are called the NP Weyl-Lanczos equations. [1]

In [3] it has been pointed out that if the Lanczos scalars in the NP Weyl-Lanczos equations are replaced by the spin coefficients according to the scheme

$$
\begin{array}{llll}
L_0 = \dfrac{\kappa}{2}, & L_4 = \dfrac{\sigma}{2} \\
L_1 = \dfrac{\rho}{6}, & L_5 = \dfrac{\tau}{6} \\
L_2 = -\dfrac{\pi}{6}, & L_6 = -\dfrac{\mu}{6} \\
L_3 = -\dfrac{\lambda}{2}, & L_7 = -\dfrac{\nu}{2}
\end{array} \tag{4}
$$

then the resulting equations for the spin coefficients can be shown to be satisfied in Petrov type N spaces with a suitably chosen dyad, by virtue of the Ricci equations of the NP-formalism. Furthermore it is shown that the *different*

[1] Unfortunately these equations contain some misprints in both [3] and [14]; in [8] the equations (1) subject to the Lanczos differential gauge $\nabla^{AA'} L_{ABCA'} = 0$ are quoted in NP-formalism and it should be noted that the equations occur there with the opposite sign compared to above.

replacement

$$
\begin{aligned}
L_0 &= \kappa\,, & L_4 &= \sigma \\
L_1 &= \frac{\rho}{3}\,, & L_5 &= \frac{\tau}{3} \\
L_2 &= -\frac{\pi}{3}\,, & L_6 &= -\frac{\mu}{3} \\
L_3 &= -\lambda\,, & L_7 &= -\nu, \qquad (5)
\end{aligned}
$$

again because of the Ricci equations, satisfies the Weyl-Lanczos equations of a type III spacetime, in a suitably chosen dyad.

These results can both be extended to Petrov type 0 spaces, but neither of these results — nor any obvious modification — is applicable to any other spaces.

In Section 2 we show how the results in [3] are actually consequences of a simple spinor ansatz

$$L_{ABC}{}^{A'} = \nabla_{(A}{}^{A'}\Big(g\, o_B \iota_{C)}\Big)$$

where g is an arbitrary function, and a very simple spinor calculation. We show that this also makes it clear why the applicability of this ansatz is restricted to spacetimes with very special Weyl spinors.

In [14] a 'generalised Weyl-Lanczos equation'

$$\Psi_{ABCD} = 2f\nabla_{(A}{}^{A'}L_{BCD)A'} \tag{6}$$

was proposed for type D vacuum spacetimes. Possible choices are $f = \Psi_2^{\frac{2}{3}}$ and $f = \Psi_2^{\frac{1}{3}}$; for these choices it was shown that the Lanczos scalars can be chosen as f^{-1} times linear combinations of spin coefficients, in an analogous manner as for the type III, N and 0 potentials found in [3]. In Section 3 it is shown how these 'generalised Lanczos potentials' can also be presented as a simple, direct ansatz via a spinor dyad. It is then obvious that this particular algorithm will *only* work for Petrov type D, vacuum spaces (or insignificant non-vacuum generalizations).

Although the particular identifications between the Lanczos scalars and the spin coefficients presented in [3] and [14] seem to be simply mathematical curiosities which are incapable of any significant direct generalization, it is important to note that there are fundamental structural links between Lanczos potentials and spin coefficients, as we discuss in Section 4. There we demonstrate such a link for a subclass of Kerr-Schild spacetimes, and moreover show the relationship to curvature-free asymmetric connections. More precisely, given a spinor dyad (o^A, ι^A), the spin coefficients of this dyad are given by the components of the spinor

$$\gamma_{CBAA'} = \iota_C \nabla_{AA'} o_B - o_C \nabla_{AA'} \iota_B.$$

We prove that in a Kerr-Schild spacetime with metric $g_{ab} = \eta_{ab} + f l_a l_b$ where η_{ab} is a flat metric and l^a is a geodesic, shear-free null-vector, there exists a

spinor dyad (o^A, ι^A) such that $L_{ABCA'} = \frac{1}{2}\gamma_{(ABC)A'}$ is a Lanczos potential of the Weyl spinor.

2 LANCZOS POTENTIALS FOR TYPE III, N AND 0 WEYL SPINORS

Of course, the first thought of how to construct an algorithm for the Lanczos potential would be to think in terms of derivatives of the metric since, in linearized theory we can obtain the Weyl tensor as a certain combination of the metrics second derivatives and consequently we can obtain a 'linearized Lanczos potential' expressed in the first partial derivatives of the metric components in a suitable coordinate system. However, in the full, non-linear theory the corresponding equation would contain other terms and so, the translation back to covariant derivatives would be difficult, even impossible if the remainder terms failed to cancel.

Instead of the metric, one could next think in terms of derivatives of the tetrad vectors. However, it is clear that when working with the Lanczos potential (in four dimensional spacetimes) the spinor structure is much simpler than the tensor structure; one needs only to compare the simple defining equation (1) in spinors with the equivalent complicated defining equation (see [13], [9]) in tensors. Therefore we will try to build our Lanczos potential from the spinor dyad (o^A, ι^A) where $o_A \iota^A = 1$.

One of the simplest constructions one could think of is

$$L_{ABC}{}^{A'} = \nabla_{(A}{}^{A'}\Big(g\, o_B \iota_{C)}\Big) \tag{7}$$

where g is an arbitrary function of the points in spacetime. Expanding the Weyl spinor in this dyad gives

$$\begin{aligned}\Psi_{ABCD} \quad = \quad & \Psi_0 \iota_A \iota_B \iota_C \iota_D - 4\Psi_1 o_{(A}\iota_B\iota_C\iota_{D)} + 6\Psi_2 o_{(A} o_B \iota_C \iota_{D)} \\ & -4\Psi_3 o_{(A} o_B o_C \iota_{D)} + \Psi_4 o_A o_B o_C o_D.\end{aligned}$$

Thus, we obtain from (7)

$$\begin{aligned}2\nabla_{(A}{}^{A'} L_{BCD)A'} \quad = \quad & -2\nabla_{A'(D}\nabla_A{}^{A'}\big(g\, o_B \iota_{C)}\big) = 2g\Psi_{(ABC}{}^{E}\big(o_{D)}\iota_E + \iota_{D)} o_E\big) \\ = \quad & 2g\big(\Psi_0 \iota_A\iota_B\iota_C\iota_D - 3\Psi_1 o_{(A}\iota_B\iota_C\iota_{D)} + \Psi_1 o_{(A}\iota_B\iota_C\iota_{D)} \\ & +3\Psi_2 o_{(A} o_B \iota_C \iota_{D)} - 3\Psi_2 o_{(A}\iota_B\iota_C o_{D)} + 3\Psi_3 o_{(A} o_B \iota_C o_{D)} \\ & -\Psi_3 o_{(A} o_B o_C \iota_{D)} - \Psi_4 o_A o_B o_C o_D\big) \\ = \quad & 2g\Psi_0 \iota_A\iota_B\iota_C\iota_D - 4g\Psi_1 o_{(A}\iota_B\iota_C\iota_{D)} + 4g\Psi_3 o_{(A} o_B o_C \iota_{D)} \\ & -2g\Psi_4 o_A o_B o_C o_D.\end{aligned} \tag{8}$$

From this calculation we can draw a number of conclusions:

If Ψ_{ABCD} is type 0, then clearly $2\nabla_{(A}{}^{A'}L_{BCD)A'} = 0 = \Psi_{ABCD}$ for all g so $L_{ABCA'}$, as given by (7), is a Lanczos potential for the Weyl spinor for any choice of g.

If Ψ_{ABCD} is type N, then we can choose o^A as the principal spinor of Ψ_{ABCD}. Then $\Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = 0$, so

$$2\nabla_{(A}{}^{A'}L_{BCD)A'} = -2g\Psi_4 o_A o_B o_C o_D = -2g\Psi_{ABCD}.$$

This means that $L_{ABCA'}$ is a Lanczos potential for the Weyl spinor if and only if $g = -\frac{1}{2}$. Note that this holds for any choice of ι^A, as long as $o_A\iota^A = 1$. For this choice, $L_{ABCA'}$ can easily be seen to coincide with the potential (4) originally found in [3].

If Ψ_{ABCD} is type III, then we can choose o^A as the repeated principal spinor of Ψ_{ABCD} and ι^A as the other principal spinor so that $\Psi_0 = \Psi_1 = \Psi_2 = \Psi_4 = 0$. Hence,

$$2\nabla_{(A}{}^{A'}L_{BCD)A'} = 4g\Psi_3 o_{(A} o_B o_C \iota_{D)} = -g\Psi_{ABCD}$$

so $L_{ABCA'}$ as given by (7) is a Lanczos potential for the Weyl spinor if and only if $g = -1$. This choice is easily seen to coincide with (5) originally found in [3].

If Ψ_{ABCD} is type D, then we can choose o^A and ι^A as the repeated principal spinors of Ψ_{ABCD} so that $\Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0$. Thus, in this case

$$2\nabla_{(A}{}^{A'}L_{BCD)A'} = 0$$

for all functions g. Therefore we cannot use $L_{ABCA'}$ given by (7) as a Lanczos potential for the Weyl spinor in type D. We can however use it as a gauge transformation i.e., if we know a Lanczos potential for a type D Weyl spinor, then we can add $L_{ABCA'}$ given by (7) to it, and the sum will still be a Lanczos potential of the Weyl spinor.

For other, more general spacetimes, it is easily seen that $L_{ABCA'}$ as given by (7) will not be a Lanczos potential of Ψ_{ABCD} because the crucial Ψ_2-component vanishes for any choice of g.

3 A ‘GENERALIZED LANCZOS POTENTIAL’ FOR WEYL SPINORS OF EMPTY, TYPE D SPACETIMES

In [14] a ‘generalized Weyl-Lanczos equation’

$$\Psi_{ABCD} = 2f\nabla_{(A}{}^{A'}L_{BCD)A'} \tag{9}$$

is proposed for a type D vacuum spacetime. Some particular solutions for $L_{ABCA'}$ are found for the choices $f = \Psi_2^{\frac{2}{3}}$ and $f = \Psi_2^{\frac{1}{3}}$. These ‘generalized

Lanczos potentials' can be found after some extensive NP calculations using both the Ricci equations and the Bianchi equations.

For $f = \Psi_2^{\frac{1}{3}}$ it is shown that a solution is given by

$$
\begin{aligned}
L_i &= 0\ , \quad i = 0,3,4,7 \\
L_1 &= -\frac{\varepsilon}{3} f^{-1}\ , \quad L_5 \quad = \quad -\frac{\beta}{3} f^{-1} \\
L_2 &= -\frac{1}{3}(\pi + \alpha) f^{-1}\ , \quad L_6 \quad = \quad -\frac{1}{3}(\mu + \gamma) f^{-1}.
\end{aligned} \tag{10}
$$

However it is easy to see that these expressions can be obtained in a similar way as in Section 2 i.e., $L_{ABCA'}$ is given simply by

$$L_{ABCA'} = -f^{-1} o_{(A} \nabla_{B|A'|} \iota_{C)}. \tag{11}$$

Here o^A and ι^A are the principal spinors of Ψ_{ABCD} (normalized so that $o_A \iota^A = 1$). Obviously an alternative is

$$L_{ABCA'} = f^{-1} \iota_{(A} \nabla_{B|A'|} o_{C)}. \tag{12}$$

For the case $f = \Psi_2^{\frac{2}{3}}$ we can obtain the form for $L_{ABCA'}$ given in [14] from

$$L_{ABCA'} = 3f^{-1} \Big(\iota_{(A} \nabla_{B|A'|} o_{C)} - \iota^D o_{(A} \iota_B \nabla_{C)A'} o_D \Big) \tag{13}$$

or alternatively

$$L_{ABCA'} = -3f^{-1} \Big(o_{(A} \nabla_{B|A'|} \iota_{C)} + o^D o_{(A} \iota_B \nabla_{C)A'} \iota_D \Big) \tag{14}$$

By calculations similar to those in the previous section it can be shown that these Lanczos spinors, with the appropriate choice of f, satisfy (9).

4 THE SPIN-COEFFICIENTS AS LANCZOS SCALARS

In [3] it has been conjectured that in general the dyad components of a Lanczos potential will be given by Lanczos scalars that are linear combinations of the spin-coefficients. In support of this, the authors of [3] have found other different identifications [4] between Lanczos scalars and the spin coefficients which 'work' in a similar way for certain other very specialised spaces e.g., for the Schwarzschild metric in a suitably chosen dyad

$$L_i = 0,\ i \neq 1,6\ , \quad L_1 = L_6 = \frac{2}{3}\varepsilon$$

It seems to us that these very special results in [3] and [4] are not part of a larger picture, but are simply mathematical coincidences in very special spaces where there is comparatively little structure. We note that the result for type N is independent of our choice for ι^A whereas in type III we have to choose ι^A as our second principal spinor of Ψ_{ABCD}. In both of these cases only properly weighted spin coefficients are used in the identification with the Lanczos scalars i.e., the results still have spin-boost freedom, but in some of the other spaces in [4], such as Schwarzschild, the Lanczos scalars are identified with non-weighted spin coefficients, so that there is no remaining dyad freedom. Most crucially, a 'relationship' between Lanczos scalars and spin coefficients which needs to change depending on Petrov type, and even subtypes, appears to be more of a curiosity than a manifestation of some genuine deep mathematical structure.

However, we believe that there are structural relationships still to be fully understood and exploited between the Lanczos potential and spin coefficients as we will explain below.

Noting that the successful choices for $L_{ABCA'}$ in Section 2, can be written

$$L_{ABCA'} = c\Big(o_{(A}\nabla_{B|A'|}\iota_{C)} + \iota_{(A}\nabla_{B|A'|}o_{C)}\Big)$$

where the constant c varies with Petrov type, therefore suggests that we try some variations e.g.,

$$L_{ABCA'} = \frac{1}{2}\Big(\iota_{(A}\nabla_{B|A'|}o_{C)} - o_{(A}\nabla_{B|A'|}\iota_{C)}\Big). \tag{15}$$

For future convenience we put

$$\gamma_{ABCA'} = \iota_A\nabla_{CA'}o_B - o_A\nabla_{CA'}\iota_B$$

so that $L_{ABCA'} = \frac{1}{2}\gamma_{(ABC)A'}$. By calculations similar to those in Section 2 we easily obtain

$$2\nabla_{(A}{}^{A'}L_{BCD)A'} = \nabla_{(A}{}^{A'}\gamma_{BCD)A'} = \Psi_{ABCD} - 2\nabla_{(A}{}^{A'}o_D\nabla_{B|A'|}\iota_{C)}. \tag{16}$$

By noting the relationship $\nabla_{(A}{}^{A'}o_D\nabla_{B|A'|}\iota_{C)} = \frac{1}{2}\gamma_{E(DA}{}^{A'}\gamma^E{}_{CB)A'}$ it follows that

$$2\nabla_{(A}{}^{A'}L_{BCD)A'} = 2\nabla_{(A}{}^{A'}\gamma_{BCD)A'} = \Psi_{ABCD} - \gamma_{E(DA}{}^{A'}\gamma^E{}_{CB)A'}. \tag{17}$$

Clearly, this choice of $L_{ABCA'}$ will be a Lanczos potential if and only if

$$\gamma_{E(DA}{}^{A'}\gamma^E{}_{CB)A'} = 0.$$

The spinor $\gamma_{ABCA'}$ is of course a familiar structure; the dyad components $\gamma_{\alpha\beta\gamma\alpha'}$ (Greek letters denote dyad indices and range from 0 to 1) of $\gamma_{ABCA'}$ yield the familiar NP spin coefficients.[2] Thus, we have the following result:

[2]Note that in [16] the indices of $\gamma_{\alpha\beta\gamma\alpha'}$ are arranged somewhat differently.

Lemma 4.1 *A Lanczos potential $L_{ABCA'}$ of the Weyl spinor can be directly equated to the spin coefficients $\gamma_{ABCA'}$ i.e., $L_{ABCA'} = \frac{1}{2}\gamma_{ABCA'}$ if and only if*

$$\gamma_{E(DA}{}^{A'}\gamma^{E}{}_{CB)A'} = 0. \tag{18}$$

The link in differential structure between equations (1) and (17) has been commented on also by Bonanos [7].

We can write the condition (18) out in full in the familiar spin coefficient notation as

$$\begin{aligned}
0 &= \sigma\varepsilon - \kappa\beta \\
0 &= -\kappa(\mu+\gamma) + \sigma(\pi+\alpha) - \rho\beta + \tau\varepsilon \\
0 &= -\kappa\nu + \sigma\lambda - \rho\mu + \tau\pi - \rho\gamma + \tau\alpha - \mu\varepsilon + \pi\beta \\
0 &= -\nu(\rho+\varepsilon) + \lambda(\tau+\beta) - \mu\alpha + \pi\gamma \\
0 &= -\nu\alpha + \lambda\gamma
\end{aligned} \tag{19}$$

It is clear, from consideration of familiar simple spacetimes in NP-formalism, that (19) does not usually hold in the familiar choices of spinor dyad e.g., vacuum type D with $\kappa = \sigma = \nu = \lambda = 0$ or N with $\kappa = \sigma = \tau = \pi = \lambda = 0$.

However, this does not mean that Lanczos potentials cannot be constructed from spin coefficients in this manner; rather it poses the question as to whether dyads can be found in which the spin coefficients satisfy (19).

Recently Lanczos potentials have been found for a class of spacetimes — Kerr-Schild spacetimes [12] in which the null-vector occurring in the metric is geodesic and shear-free. Further these Lanczos potentials have been found in the context of curvature-free, asymmetric metric connections. Recall that any metric connection $\hat{\nabla}_{AA'}$ can be written

$$\hat{\nabla}_{AA'}\xi^B = \nabla_{AA'}\xi^B + 2\Gamma_C{}^B{}_{AA'}\xi^C \tag{20}$$

where $\Gamma_{CBAA'} = \Gamma_{(CB)AA'}$. Further, recall that a Kerr-Schild spacetime is a spacetime in which the metric g_{ab} can be written $g_{ab} = \eta_{ab} + f l_a l_b$ for some function f, where η_{ab} is a flat metric and l^a is a null-vector with respect to g_{ab}. In [2] the following theorem is proved:

Theorem 4.2 *In a Kerr-Schild spacetime the spinor*

$$\Gamma_{ABCA'} = \frac{1}{2}\nabla_{(A}{}^{B'}\left(f\xi_{B)}\xi_C\xi_{A'}\xi_{B'}\right)$$

where $l_a = \xi_A\xi_{A'}$, defines an asymmetric connection $\hat{\nabla}_{AA'}$ according to (20), with vanishing curvature tensor. Furthermore, if the null-vector l^a is geodesic and shear-free, the spinor $L_{ABCA'} = \Gamma_{(ABC)A'}$ is a Lanczos potential of the Weyl spinor.

We remark that the first part of the theorem was proved by Harnett [10] while Bergqvist [6] has proved the complete theorem in the special case of the Kerr spacetime.

It is well-known that since $\hat{\nabla}_{AA'}$ has zero curvature, there exists a normalized spinor dyad (o^A, ι^A) such that $\hat{\nabla}_{AA'} o_B = \hat{\nabla}_{AA'} \iota_B = 0$. From this we easily obtain the relation

$$\Gamma_{CBAA'} = \frac{1}{2}(\iota_C \nabla_{AA'} o_B - o_C \nabla_{AA'} \iota_B) = \frac{1}{2}\gamma_{CBAA'}$$

where $\gamma_{CBAA'}$ are the spin coefficients of the dyad (o^A, ι^A) as above. This proves the following corollary:

Corollary 4.3 *In a Kerr-Schild spacetime in which l^a is geodesic and shear-free there exists a normalized spinor dyad (o^A, ι^A) with spin coefficients $\gamma_{ABCA'}$, such that $L_{ABCA'} = \frac{1}{2}\gamma_{(ABC)A'}$, is a Lanczos potential of the Weyl spinor.*

However, it is emphasised again that neither of the elements of the spinor dyad of the above theorem need coincide with the principal spinors of the Weyl spinor or the spinor ξ_A occurring in the metric.

Whether dyads with the required properties exist for other spacetimes, or indeed for all spacetimes, is an open question which we are at presently investigating. It seems to be in this context of curvature-free asymmetric connections that investigating the links between Lanczos potentials and spin coefficients will be most useful.

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